RESIDUES OF CERTAIN SUMS OF POWERS OF INTEGERS.

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In Vol. XXXI, pages 329–333 of the *Quarterly Journal of Mathematics*, GLAISHER showed that, if p be an odd prime, and

$$H_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{(p-1)^n},$$

then, when n is not a multiple of $p - 1$, $H_n \equiv 0 \bmod p$; and when n is a multiple of $p - 1$, $H_n \equiv -1 \bmod p$. By using Fermat's Theorem we have

$$1/a^n \equiv a^{p-n-1} \bmod p,$$

so that, putting $p - n - 1 = r$, we obtain

$$H_n \equiv 1 + 2^r + 3^r + \cdots + (p-1)^r \bmod p.$$

Hence the sum of the rth powers of the first $p - 1$ integers is congruent to -1, or zero, according as r is, or is not, a multiple of $p - 1$.

These results of Glaisher, as well as some others, may be obtained in an elementary way by starting from the identity,[1]

$$p^{r+1} - p = {}_{r+1}C_1 S_r + {}_{r+1}C_2 S_{r-1} + \cdots + {}_{r+1}C_r S_1,$$

where $S_r = 1 + 2^r + 3^r + \cdots + (p-1)^r$.

By taking, successively, $r = 1, 2, 3, \cdots p - 2$, we obtain

$$S_r \equiv 0 \bmod p.$$

If $r = p - 1$, $a^r \equiv a^{p-1} \equiv 1 \bmod p$; hence $S_{p-1} \equiv p - 1 \equiv -1 \bmod p$.

Furthermore since $S_{r'} \equiv S_r \bmod p$ whenever $r' \equiv r \bmod p - 1$, the residues of this series are known for every power r.

Since $a^{2k} \equiv (p-a)^{2k} \bmod p$,

$$S_{2k} \equiv 1^{2k} + 2^{2k} + 3^{2k} + \cdots + 3^{2k} + 2^{2k} + 1^{2k};$$

hence denoting by T_r the sum

$$1^r + 2^r + 3^r + \cdots + \left(\frac{p-1}{2}\right)^r$$

we obtain

$$2T_{2k} \equiv S_{2k} \bmod p.$$

Therefore, when $2k$ is not a multiple of $p - 1$, $T_{2k} \equiv 0 \bmod p$; and when $2k$ is a multiple of $p - 1$

$$T_{2k} \equiv -\tfrac{1}{2} \bmod p.$$

The determination of the residues of T_{2k-1} may also be made to depend upon

[1] C. Smith, *Treatise on Algebra*, 5th edition, p. 404.

the original identity. If the latter be written in terms of the T's it takes the form

$$\left(\frac{p+1}{2}\right)^{r+1} - \left(\frac{p+1}{2}\right) = {}_{r+1}C_1 T_r + {}_{r+1}C_2 T_{r-1} + \cdots + {}_{r+1}C_r T_1$$

or

$${}_{r+1}C_1 T_1 + {}_{r+1}C_2 T_2 + \cdots + {}_{r+1}C_r T_r \equiv \frac{1}{2^{r+1}} - \frac{1}{2} \bmod p.$$

By letting $r = 1, 2, 3, \cdots$, successively, the residues of T_1, T_2, T_3, etc., may be computed. It will be found as shown above that for any even value of r the residue of T_r is zero. The first seven odd values of r give the following results.

$$T_1 \equiv -\frac{1}{2^3}, \qquad T_3 \equiv \frac{1}{2^6}, \qquad T_5 \equiv -\frac{1}{2^7}, \qquad T_7 \equiv \frac{17}{2^{11}},$$

$$T_9 \equiv -\frac{31}{2^{11}}, \qquad T_{11} \equiv \frac{691}{2^{14}}, \qquad T_{13} \equiv -\frac{5461}{2^{15}}, \qquad \text{each taken mod } p.$$

These results hold for any prime p.

From a formula given by Glaisher in the *Quarterly Journal,* volume XXXII, page 279, a general expression for these residues may be derived. The formula is

$$\frac{1}{2^{2i+1}} + \frac{1}{4^{2i+1}} + \cdots + \frac{1}{(p-1)^{2i+1}} \equiv (-1)^{h-i}(2^{2h-2i} - 1)\frac{B_{h-i}}{2h-2i} \quad \bmod p,$$

where $2h = p - 1$ and B_{h-i} is the Bernoulli number of rank $h - i$.

Remembering that $1/a^{2i+1} \equiv a^{p-2i-2} \bmod p$, and taking out $1/2^{2i+1}$ from the left member, we get

$$\frac{1}{2^{2i+1}}\left\{1 + 2^{p-2i-2} + 3^{p-2i-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2i-2}\right\}$$
$$\equiv (-1)^{h-i} \cdot (2^{2h-2i} - 1)\frac{B_{h-i}}{2h-2i} \quad \bmod p.$$

Letting $h - i = k$, or $p - 2i - 2 = 2k - 1$, and $2i + 1 = p - 2k$, we get

$$T_{2k-1} \equiv (-1)^k \cdot 2^{p-2k}(2^{2k} - 1) \cdot \frac{B_k}{2k} \equiv (-1)^k \frac{2^{2k}-1}{2^{2k}} \cdot \frac{B_k}{k} \quad \bmod p.$$

This formula holds for $k = 1, 2, \cdots, \frac{1}{2}(p-1)$, but for $k = \frac{1}{2}(p-1)$ the denominator of B_k is divisible by p, so that it should be written

$$T_{p-2} \equiv (-1)^{\frac{1}{2}(p-1)} \cdot \frac{2^{p-1}-1}{2^{p-1}p} \cdot \frac{pB_{\frac{1}{2}(p-1)}}{\frac{1}{2}(p-1)}$$
$$\equiv (-1)^{\frac{1}{2}(p+1)} \cdot 2 \cdot \frac{2^{p-1}-1}{p} \cdot pB_{\frac{1}{2}(p-1)} \quad \bmod p.$$

But $pB_{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{2}(p-1)} \bmod p$.[1] Hence

$$T_{p-2} \equiv -2 \cdot \frac{2^{p-1}-1}{p} \quad \bmod p.$$

It follows from the above formulas that for no value of k is T_{2k-1} congruent to zero for every odd prime p, though there are, in general, special primes for each value of $k > 3$ for which T_{2k-1} will be congruent to zero.

The residue of T_{p-2} is seen to depend upon the residue of $(2^{p-1} - 1)/p$. The vanishing of this residue has been shown by Wieferich to be a necessary condition in order that Fermat's equation $x^p + y^p = z^p$ shall be solvable in integers prime to p. However, as stated on page 9 of *L'Intermédiaire des Mathématiciens*, January, 1914, this residue does vanish for $p = 1,093$, though for no other primes less than 2,000.

BOOK REVIEWS.

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Plane and Solid Geometry. By WALTER BURTON FORD, Junior Professor of Mathematics in the University of Michigan, and CHARLES AMMERMAN, of the Wm. McKinley High School, St. Louis. Edited by EARLE RAYMOND HEDRICK. The Macmillan Company, New York, 1913. ix + 321 + xxxiii pages. $1.25. Plane and Solid in separate volumes $0.80 each.

Another interesting text-book of the series edited by Earle Raymond Hedrick, of the University of Missouri, has made its appearance; an attractive volume in a neat brown dress. The work, as we would expect, under Professor Hedrick's editorship, departs from the class of text-books written by scientists not in touch with life. The Euclidean division into books has been abandoned for the modern division into chapters.

The pupil is brought into touch with the subject matter of geometry in the introduction, covering 33 pages, which is divided into three parts: (I) Drawing simple figures, (II) the principal ideas used in geometry, (III) statements for reference. This division of the book contains 111 exercises. Squared paper is used to estimate areas and construct designs.

The arrangement of the subject matter of Plane Geometry in the first five chapters follows that of the usual modern standard text. Logical and original thinking are sure to be developed through the "judicious fusion of theoretical and applied work," here presented. There are approximately 750 exercises in these five chapters. Dr. Frank M. McMurry said, in a paper on "How to Study" given at the Minnesota State Teachers' Association last fall: "Do we in school know how to arrange the work so the pupil will learn to select? Teachers object to neglecting any part of the text. They dwell upon all parts with equal

[1] *Quarterly Journal*, Vol. XXXII, p. 273.